STEADY MOTION OF A STRATIFIED FLUID OVER A ROUGH BOTTOM

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The problem of steady internal gravity waves in stratified flow over a rough bottom is of great significance in connection with the investigation of wave processes in the ocean and the atmosphere. There are two common methods of solving this problem in the linear approximation. The first consists in the exact numerical solution of the linearized system of equations of hydrodynamics [1, 2], the second in replacing the function describing the shape of the rough bottom either by a function having a simple form (e.g., a hemispherical shape [3]) or by a system of point sources taken with a certain weighting [1, 4]; as a result, for particular cases of the Brunt–Väisälä frequency distribution N(z) (N(z) is usually assumed to be constant [3, 4]) the problem can be solved analytically. The shortcomings of the first method includes the boundedness of the region of space in which the problem can be solved numerically, while in investigating the problem by the second method it is not possible to estimate the limits of applicability of the approximations. Accordingly, there is interest in solving the problem using the Green' function of the internal wave equation and also its asymptotic form [5-7], which makes it possible not only to investigate the problem numerically but also to employ various approximations to simplify the solution.

It is proposed to consider the problem of stratified flow with an arbitrary Brunt–Väisälä frequency distribution over a rough bottom when the height of the underwater obstacle is assumed to be small as compared with the thickness of the fluid layer. The free surface z = 0 is replaced by a rigid roof, and the Cartesian coordinate system is so chosen that the plane x, y lies on the horizontal surface of the bottom. When $x \to -\infty$ the flow is asymptotically one-dimensional with constant velocity V along the x axis. The flow is assumed to be weakly stratified, i.e., the internal Froude number $Fr = V/N_* h_*$ (N_* is the characteristic Brunt–Väisälä frequency, h_* is the height of the obstacle) is greater than unity. Physically, this means that the fluid particles in the undisturbed flow possess sufficient kinetic energy to rise to the height of the obstacle, i.e., the pattern of the bottom trajectories must have qualitatively the same form as in the case of a homogeneous fluid [3].

The vertical component w of the internal wave (IW) velocity satisfies the following IW equation, which can be obtained from the linearized system of equations of hydrodynamics in the Boussinesq approximation [5]:

$$L_* w = 0,$$
$$L_* = \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + N^2(z)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

On the bottom, the shape of whose surface is described by the function $Z(x, y) = -h^0 + h(x, y)$, the no-flow condition $\mathbf{u}$ grad Z = 0 is satisfied ($\mathbf{u} = (V + u_1, u_2\ w)$ is the IW velocity vector, (u_1, u_2) are the horizontal components of the IW velocity). After linearization, on the assumption that $h(x, y) \ll h^0$, the boundary condition is transferred to the bottom $z = -h^0$, and has the form [2] $w = -V\partial h/\partial x \equiv f(x, y)$, $z = -h^0$.

Borovitkov et al. [5] considered the function $G_*(x, y, z, z_0, t)$, which is the solution of the problem

$$L_* G_* = Q\theta(t)\,\delta''_{tt}(x - Vt)\,\delta(y)\,\delta'(z - z_0),$$
$$G_* = 0, \quad z = 0, -h^0.$$

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Here, $\theta(t) = 0$ when $t < 0$; $\theta(t) = 1$ when $t \geq 0$; z_0 is the depth of the point mass source switched on at $t = 0$; and Q is its strength. In this case the function G_* describes the vertical IW velocity field. We write the limit of G_* for $t \to \infty$ and $\xi = x - VT$ in the form:

$$G(\xi, y, z, z_0) = \lim_{t\to\infty} G_*(x, y, z, z_0, t),$$

where

$$G = \sum_{n=1}^{\infty} G_n;\ G_n = -I_n^0 (\xi < 0);\ G_n = I_n^0 + I_n^- + I_n^+ (\xi > 0);$$

$$I_n^{\pm} = \frac{Q}{4\pi i} \int_{-\infty}^{\infty} \exp(\mp i\mu_n(\nu)\xi - i\nu y) A_n(\nu, z, z_0)\, d\nu;$$

$$I_n^0 = \frac{Q}{4\pi} \int_{-\infty}^{\infty} \exp(-\lambda_n(\nu)|\xi| - i\nu y) B_n(\nu, z, z_0)\, d\nu;$$

$$A_n(\nu, z, z_0) = \frac{\mu_n^3(\nu) V^2}{\mu_n^2(\nu) + \nu^2} \left(\frac{\mu_n(\nu)\mu_n'(\nu)}{\nu} + 1\right) \varphi_n(z, \nu) \frac{\partial \varphi_n(z_0, \nu)}{\partial z_0};$$

$$B_n(\nu, z, z_0) = \frac{\lambda_n^3(\nu) V^2}{\lambda_n^2(\nu) - \nu^2} \left(\frac{\lambda_n(\nu)\lambda_n'(\nu)}{\nu} - 1\right) \psi_n(z, \nu) \frac{\partial \psi_n(z_0, \nu)}{\partial z_0}.$$

Here $\mu_n(\nu)$, $\lambda_n(\nu)$, $\phi_n(z, \nu)$ and $\psi_n(z, \nu)$ are the eigenvalues and eigenfunctions of the corresponding problems

$$\frac{\partial^2 \varphi_n}{\partial z^2} + [\mu_n^2(\nu) + \nu^2] \left[\frac{N^2(z)}{V^2 \mu_n^2(\nu)} - 1\right] \varphi_n = 0,$$

$$\frac{\partial^2 \psi_n}{\partial z^2} + [\lambda_n^2(\nu) - \nu^2] \left[\frac{N^2(z)}{V^2 \lambda_n^2(\nu)} + 1\right] \psi_n = 0,$$

$$\varphi_n = \psi_n = 0,\ z = 0,\ -h^0.$$

The function $G(\xi, y, z, z_0)$, which describes the vertical velocity field of the steady internal waves from the source in the stratified flow, satisfies the equation

$$LG = V^2 \delta''(\xi)\delta(y)\delta'(z - z_0),\quad L = V^2 \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + N^2(z)\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial y^2}\right).$$

Since we are interested in the steady flow of a stratified fluid over a rough bottom, the vertical component w of the IW velocity must satisfy the equations.

$$Lw = 0; \tag{1}$$

$$w = 0,\ z = 0; \tag{2}$$

$$w = f(\xi, y),\ z = -h^0. \tag{3}$$

We now denote ξ by x; then the function w satisfying Eq. (1) and boundary conditions (2), (3) can be represented in the form

$$w = \frac{1}{Q} \iint_{\Omega} G(x - x', y - y', z, -h^0) f(x', y')\, dx'\, dy', \tag{4}$$

where Ω is the domain, and $f(x, y) \neq 0$. The solution thus obtained is a sum of triple quadratures, which complicates both the numerical calculation of the function w and its qualitative analysis; accordingly, in what follows we will use the asymptotic forms of the function $G(x, y, z, z_0)$ constructed in [6, 7]. To be specific, we will consider the obstacle shape from [2]; then a vertical cut of the function h(x, y) perpendicular to the x axis will be a semiellipse and, folowing the notation of [2], we can represent the function h(x, y) in the form $h(x, y) = H(x) \times W^{-1}(x)(W^2(x) - y^2)^{1/2}$ when $y \leq W(x)$ and $h(x, y) = 0$ when $y > W(x)$, where $H(x) = 3/2\ Dx_*^2 (1 + x_*^4)^{-1}$, $W(x) = H(x)$ when $x \leq W$, $W(x) = 3/4\ Dx_*^{0.4}$ when $x > X$, $x_* = x/X$, X = 25 m, and D = 13 m. We have $h^0 = 50$ m, and the maximum height of the obstacle is $0.2h^0$ (Fig. 1). In this case it is possible to integrate with respect to the variable y' and write the expression for the function w in the form:

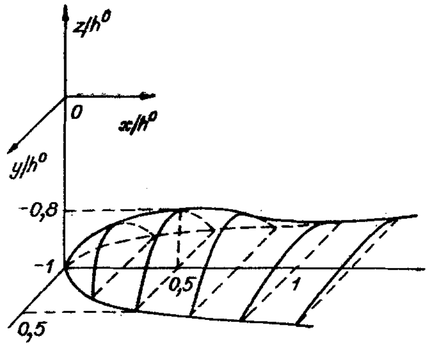

Fig. 1

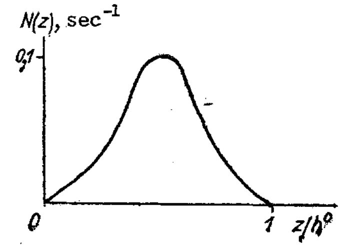

Fig. 2

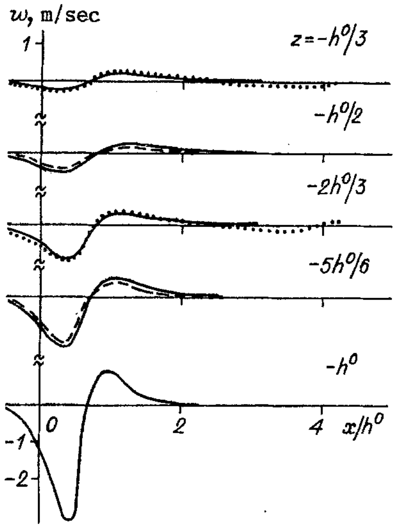

Fig. 3

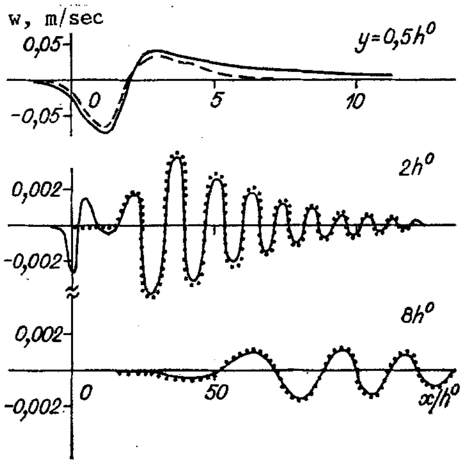

Fig. 4

$$w=\sum_{n=1}^{\infty} w_n,\quad w_n = w_n^- + w_n^+, \tag{5}$$

$$w_n^- = -\frac{V}{Q}\int_0^x dx' \int_0^\infty A_n(\nu, z, -h^0)\sin[\mu_n(\nu)(x-x')]\cos(\nu y)P(\nu, x')\,d\nu,$$

$$w_n^+ = -\frac{V\,\mathrm{sign}\,(x-x')}{2Q}\int_0^\infty dx' \int_0^\infty B_n(\nu, z, -h^0)\times$$

$$\times\exp(-\lambda_n(\nu)\,|x-x'|)\cos(\nu y)P(\nu, x')\,d\nu,$$

$$P(\nu, x) = \frac{J_1(\tau)}{\tau}[W(x)H'(x) - H(x)W'(x)] + H(x)W(x)J_0(\tau),\quad \tau = \nu W(x)$$

($J_0(x)$ and $J_1(x)$ are zeroth-order and first-order Bessel functions). The results of numerically calculating the function w (continuous curve) for the distribution N(z) taken from [2] (Fig. 2) are presented in Figs. 3 and 4; the remaining parameters have the following values: $Q = 1\ m^3/sec$, $V = 5$ m/sec. For Fig. 3 $y = 0$, for Fig. 4 $z = -0.5h^0$. The broken curve in Fig. 3 represents the calculation of the vertical velocity for y = 0 taken from [2]. We note that all the characteristics of our numerical results coincide with those indicated in [2]: the vertical velocity w_0 rapidly decreases with decrease in the depth z and when $z = -0.5h^0$ is only about 15% of the value of w on the bottom ($z = -h^0$), which coincides with the function f(x, y). The small differences in the spatial structures of the two solutions are the result of the fact that in [2] the Boussinesq approximation was not used in the numerical solution of the problem and the surface of the fluid at z = 0 was assumed to be free.

We will now consider the function w in the region adjacent to the rough bottom. The near-field asymptotic expansion of the function $G(x, y, z, z_0)$ has the form [7]:

$$G(x, y, z, z_0) \approx \frac{Q}{4\pi}\left\{\frac{z_-}{(\rho^2+z_-^2)^{3/2}} + \frac{z_+}{(\rho^2+z_+^2)^{3/2}} - \right.$$

$$-\sum_{m=1}^{\infty}\left[\frac{2mh^0 - z_-}{\left(\rho^2+\left(2mh^0-z_-\right)^2\right)^{3/2}} - \frac{2mh^0 + z_-}{\left(\rho^2+\left(2mh^0+z_-\right)^2\right)^{3/2}} + \right. \tag{6}$$

$$\left. + \frac{2mh^0 - z_+}{\left(\rho^2+\left(2mh^0-z_+\right)^2\right)^{3/2}} - \frac{2mh^0 + z_+}{\left(\rho^2+\left(2mh^0+z_+\right)^2\right)^{3/2}} \right]\Bigg\} \equiv b(x, y, z, z_0)$$

$$(\rho^2 = x^2 + y^2,\ z_- = z - z_0,\ z_+ = z + z_0).$$

The function $b(x, y, z, z_0)$ is a sum of terms each of which is a derivative with respect to z of the fundamental solution of the Laplace equation and represents the field created by a source at the depth $z_m = \pm z_0 + 2mh^0$ ($m = 0, \pm 1, \pm 2, \ldots$) in a homogeneous fluid. The series (6) converges rapidly for large m, since the m-th term decreases as m^{-3}, and to achieve an accuracy of the order of one percent it is necessary to sum not more than ten terms of the series. Replacing the function $G(x, y, z, z_0)$ in (4) by $b(x, y, z, z_0)$, we obtain an expression for w in the region adjacent to the rough bottom:

$$w \approx \frac{1}{Q}\iint_{\Omega} b(x - x', y - y', z, -h^0) f(x', y')\,dx'\,dy' \equiv S(x, y, z). \tag{7}$$

The function S satisfies the Laplace equation and boundary conditions (2), (3), i.e., describes the flow of a homogeneous fluid over the rough bottom. Figures 3 and 4 give the results of calculating the function S (Broken curve). As the numerical calculations show, at distances of the order of h^0 from the rough bottom the flow is almost potential and depends only on the geometry of the problem. However, with increasing distance from the rough bottom it is necessary to take into account the stratification of the fluid. By means of the near-field asymptotic expansion of the function G it is also possible to determine the corrections to the potential flow needed to take stratification into account. For this purpose we represent $b(x, y, z, z_0)$ in the form [8]:

$$b(x, y, z, z_0) = \sum_{n=1}^{\infty} K_n(x, y, z, z_0),$$

$$K_n(x, y, z, z_0) = \frac{nQ}{(h^0)^2} K_0\left(\frac{\pi n \rho}{h^0}\right) \sin\left(\frac{\pi n z}{h^0}\right) \cos\left(\frac{\pi n z_0}{h^0}\right)$$

($K_0(x)$ is a zeroth-order Macdonald function). In [7] it was shown that K_n is the near-field asymptotic expansion of the individual mode G_n, and as the mode number n increases the function K_n approaches G_n with ever greater accuracy. We now represent w in the form:

$$w = S + \sum_{n=1}^{\infty} \Delta_n, \tag{8}$$

$$\Delta_n = w_n - \frac{1}{Q}\iint_{\Omega} K_n(x - x', y - y', z, -h^0) f(x', y')\,dx'\,dy'.$$

Then the procedure for calculating the corrections to the potential flow at a point x, y, z in order to take stratification into account is as follows: (1) the function S is calculated from (7), (2) then the term Δ_1 is determined; if $\Delta_1 \ll s$, then the next terms in (8) need not be found; (3) if Δ_1 is comparable with S, then the term Δ_2 is calculated; (4) if $\Delta_2 \ll S + \Delta_1$, then the summation in (8) is interrupted, and so on. The procedure described makes it possible not only to calculate accurately the IW near field but also to estimate the error of the asymptotic expressions. The results of calculating S (chain curve), $S + \Delta_1$ (broken curve), $S + \Delta_1 + \Delta_2$ (dotted curve), and w (continuous curve) for $z = -0.5h^0$, $y = 0.5h^0$ are reproduced in Fig. 5. As the numerical calculations show, at distances from the rought bottom of the order of the thickness of the fluid layer taking two corrections into account gives almost the exact value of the field, which makes it unnecessary to calculate the entire sum (5).

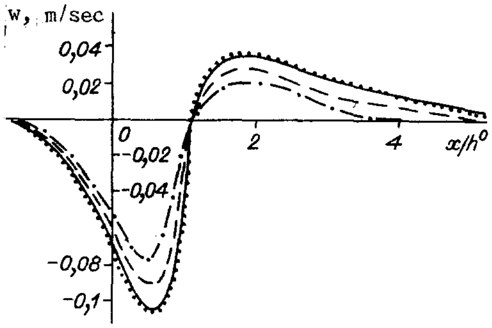

Fig. 5

At large distances from the rough bottom the IW field splits into individual modes; moreover, at large distances in the case of weak stratification it is possible to disregard the exact shape of the obstacle and replace it by a suitable system of sources (a widely used method for investigating stratified flow over various bodies [1, 9]). In the present case the bottom roughness takes the form of a single convex obstacle; therefore the exact boundary condition on the bottom can be replaced by

$$w = T(x, y),\ z = -h^0,\ T(x, y) = \Gamma\delta(y)[\delta(x_+) - \delta(x_-)],$$
$$\Gamma = \frac{1}{2}\iint\limits_{\Omega} |f(x, y)|\,dx\,dy,$$

where $x_\pm$ are the coordinates of the source and sink determined by the geometry of the problem, and since the function H(x) has a single maximum, the values of $x_\pm$ can be found from the equation $H''(x) = 0$: $x_+ \approx h^0/4$, $x_- \approx h^0$. Moreover, at large distances from the source it is possible to replace each mode G_n by its asymptotic form [6]:

$$G_n(x, y, z, z_0) \approx \frac{Q}{2\pi}\frac{A_n(\nu_*, z, z_0)}{\sqrt{2\mu_n''(\nu_*)\,x}}\,\sigma^{1/4}\mathrm{Ai}'(\sigma) \equiv$$
$$\equiv R(x, y, z, z_0),\quad \sigma = \left[\frac{3}{2}(\mu_n(\nu_*)\,x - \nu_* y)\right]^{2/3}.$$

Here, Ai'(x) is the derivative of the Airy function; and ν_* is the root of the equation $\mu_n(\nu) = y/x$. As a result, the field of the individual mode at large distances from the rough bottom can be represented as

$$w_n \approx \frac{\Gamma}{Q}[R(x_+, y, z, -h^0) - R(x_-, y, z, -h^0)]. \tag{9}$$

The results of the calculations carried out using (9) are represented by a dotted curve in Fig. 4. The numerical calculations show that at distances greater than $10h^0$ the use of the asymptotic form of the function G and the replacement of the obstacle by a suitable system of sources make it possible to calculate the IW field without resorting to clumsy calculations based on (5).

Thus, the use of the Green's function of the internal wave equation makes it possible not only to solve accurately the problem of steady flow of a stratified fluid over a rough bottom but also, on the basis of asymptotic representations of the Green's function, to investigate efficiently the internal wave fields.

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